Detailed definition of ASMs

- Part 1: Abstract states and update sets
- Part 2: Mathematical Logic
- Part 3: Transition rules and runs of ASMs
- Part 4: The reserve of ASMs

Part 1

Abstract states and update sets

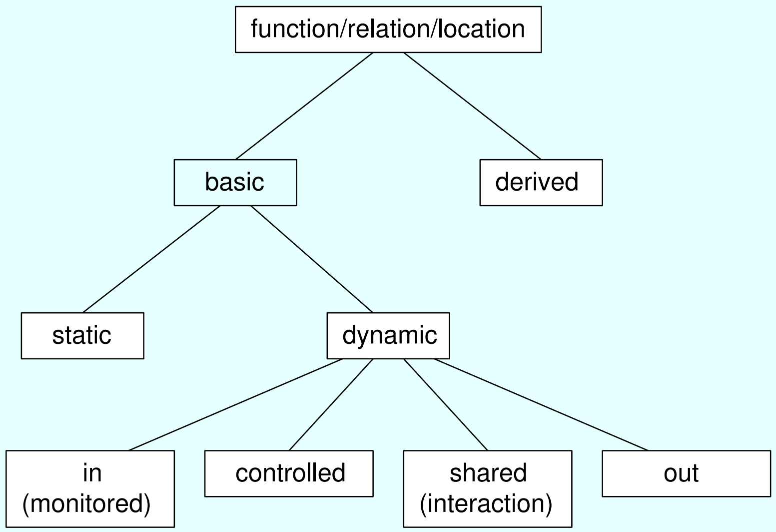
Signatures

Definition. A *signature* Σ is a finite collection of function names.

- Each function name f has an *arity*, a non-negative integer.
- Nullary function names are called *constants*.
- Function names can be *static* or *dynamic*.
- Every ASM signature contains the static constants $undef$, $true$, $false$.

Signatures are also called *vocabularies*.

Classification of functions



States

Definition. A *state* $\mathfrak{A}$ for the signature Σ is a non-empty set X, the *superuniverse* of $\mathfrak{A}$, together with an *interpretation* $f^{\mathfrak{A}}$ of each function name f of Σ.

- If f is an n-ary function name of Σ, then $f^{\mathfrak{A}}: X^n \to X$.
- If c is a constant of Σ, then $c^{\mathfrak{A}} \in X$.
- The superuniverse X of the state $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$.

- The superuniverse is also called the *base set* of the state.
- The *elements* of a state are the elements of the superuniverse.

States (continued)

- The interpretations of $undef$, $true$, $false$ are pairwise different.
- The constant $undef$ represents an undetermined object.
- The *domain* of an n-ary function name f in $\mathfrak{A}$ is the set of all n-tuples $(a_1, \ldots, a_n) \in |\mathfrak{A}|^n$ such that $f^{\mathfrak{A}}(a_1, \ldots, a_n) \neq undef^{\mathfrak{A}}$.
- A *relation* is a function that has the values $true$, $false$ or $undef$.
- We write $a \in R$ as an abbreviation for $R(a) = true$.
- The superuniverse can be divided into *subuniverses* represented by unary relations.

Locations

Definition. A *location* of $\mathfrak{A}$ is a pair

$$\boxed{(f, (a_1, \ldots, a_n))}$$

where f is an n-ary function name and $a_1, \ldots, a_n$ are elements of $\mathfrak{A}$.

- The value $f^{\mathfrak{A}}(a_1, \ldots, a_n)$ is the *content* of the location in $\mathfrak{A}$.
- The *elements* of the location are the elements of the set $\{a_1, \ldots, a_n\}$.
- We write $\mathfrak{A}(l)$ for the content of the location l in $\mathfrak{A}$.

Notation. If $l = (f, (a_1, \ldots, a_n))$ is a location for $\mathfrak{A}$ and α is a function defined on $|\mathfrak{A}|$, then $\alpha(l) = (f, (\alpha(a_1), \ldots, \alpha(a_n)))$.

Updates and update sets

Definition. An *update* for $\mathfrak{A}$ is a pair (l, v), where l is a location of $\mathfrak{A}$ and v is an element of $\mathfrak{A}$.

- The update is *trivial*, if $v = \mathfrak{A}(l)$.
- An *update set* is a set of updates.

Definition. An update set U is *consistent*, if it has no clashing updates, i.e., if for any location l and all elements v, w, if $(l, v) \in U$ and $(l, w) \in U$, then $v = w$.

Firing of updates

Definition. The result of *firing* a consistent update set U in a state $\mathfrak{A}$ is a new state $\mathfrak{A} + U$ with the same superuniverse as $\mathfrak{A}$ such that for every location l of $\mathfrak{A}$:

$$(\mathfrak{A} + U)(l) = \begin{cases} v, & \text{if } (l, v) \in U; \\ \mathfrak{A}(l), & \text{if there is no } v \text{ with } (l, v) \in U. \end{cases}$$

The state $\mathfrak{A} + U$ is called the *sequel* of $\mathfrak{A}$ with respect to U.

Homomorphisms and isomorphisms

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two states over the same signature.

Definition. A *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a function α from $|\mathfrak{A}|$ into $|\mathfrak{B}|$ such that $\alpha(\mathfrak{A}(l)) = \mathfrak{B}(\alpha(l))$ for each location l of $\mathfrak{A}$.

Definition. An *isomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ which is a ono-to-one function from $|\mathfrak{A}|$ onto $|\mathfrak{B}|$.

Lemma (Isomorphism). Let α be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. If U is a consistent update set for $\mathfrak{A}$, then $\alpha(U)$ is a consistent update set for $\mathfrak{B}$ and α is an isomorphism from $\mathfrak{A}+U$ to $\mathfrak{B}+\alpha(U)$.

Composition of update sets

$$U \oplus V = V \cup \{(l, v) \in U \mid \text{there is no } w \text{ with } (l, w) \in V\}$$

Lemma. Let U, V, W be update sets.

- $(U \oplus V) \oplus W = U \oplus (V \oplus W)$
- If U and V are consistent, then $U \oplus V$ is consistent.
- If U and V are consistent, then $\mathfrak{A} + (U \oplus V) = (\mathfrak{A} + U) + V$.

Part 2

Mathematical Logic

Terms

Let Σ be a signature.

Definition. The *terms* of Σ are syntactic expressions generated as follows:

- Variables x, y, z, ... are terms.
- Constants c of Σ are terms.
- If f is an n-ary function name of Σ, $n > 0$, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

- A term which does not contain variables is called a *ground term*.
- A term is called *static*, if it contains static function names only.
- By $t\frac{s}{x}$ we denote the result of replacing the variable x in term t everywhere by the term s (*substitution* of s for x in t).

Variable assignments

Let $\mathfrak{A}$ be a state.

Definition. A *variable assignment* for $\mathfrak{A}$ is a finite function ζ which assigns elements of $|\mathfrak{A}|$ to a finite number of variables.

- We write $\zeta[x \mapsto a]$ for the variable assignment which coincides with ζ except that it assigns the element a to the variable x:

$$\zeta[x \mapsto a](y) = \begin{cases} a, & \text{if } y = x; \\ \zeta(y), & \text{otherwise.} \end{cases}$$

- Variable assignments are also called *environments*.

Evaluation of terms

Definition. Let $\mathfrak{A}$ be a state of Σ.
Let ζ be a variable assignment for $\mathfrak{A}$.
Let t be a term of Σ such that all variables of t are defined in ζ.
The *value* $[\![t]\!]^{\mathfrak{A}}_{\zeta}$ is defined as follows:

- $[\![x]\!]^{\mathfrak{A}}_{\zeta} = \zeta(x)$
- $[\![c]\!]^{\mathfrak{A}}_{\zeta} = c^{\mathfrak{A}}$
- $[\![f(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_{\zeta} = f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_{\zeta}, \ldots, [\![t_n]\!]^{\mathfrak{A}}_{\zeta})$

Evaluation of terms (continued)

Lemma (Coincidence). If ζ and η are two variable assignments for t such that $\zeta(x) = \eta(x)$ for all variables x of t, then $[\![t]\!]^{\mathfrak{A}}_{\zeta} = [\![t]\!]^{\mathfrak{A}}_{\eta}$.

Lemma (Homomorphism). If α is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then $\alpha([\![t]\!]^{\mathfrak{A}}_{\zeta}) = [\![t]\!]^{\mathfrak{B}}_{\alpha \circ \zeta}$ for each term t.

Lemma (Substitution). Let $a = [\![s]\!]^{\mathfrak{A}}_{\zeta}$.
Then $[\![t\frac{s}{x}]\!]^{\mathfrak{A}}_{\zeta} = [\![t]\!]^{\mathfrak{A}}_{\zeta[x \mapsto a]}$.

Formulas

Let Σ be a signature.

Definition. The *formulas* of Σ are generated as follows:

- If s and t are terms of Σ, then $s = t$ is a formula.
- If φ is a formula, then $\neg\varphi$ is a formula.
- If φ and ψ are formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \to \psi)$ are formulas.
- If φ is a formula and x a variable, then $(\forall x\, \varphi)$ and $(\exists x\, \varphi)$ are formulas.

- A formula $s = t$ is called an *equation*.
- The expression $s \neq t$ is an abbreviation for $\neg(s = t)$.

Formulas (continued)

symbol	name	meaning
$\neg$	negation	not
$\wedge$	conjunction	and
$\vee$	disjunction	or (inclusive)
$\to$	implication	if-then
$\forall$	universal quantification	for all
$\exists$	existential quantification	there is

Formulas (continued)

$$\begin{array}{ll} \varphi \wedge \psi \wedge \chi & \text{stands for } ((\varphi \wedge \psi) \wedge \chi), \\ \varphi \vee \psi \vee \chi & \text{stands for } ((\varphi \vee \psi) \vee \chi), \\ \varphi \wedge \psi \to \chi & \text{stands for } ((\varphi \wedge \psi) \to \chi), \text{ etc.} \end{array}$$

- The variable x is *bound* by the quantifier $\forall$ ($\exists$) in $\forall x\, \varphi$ ($\exists x\, \varphi$).
- The *scope* of x in $\forall x\, \varphi$ ($\exists x\, \varphi$) is the formula φ.
- A variable x occurs *free* in a formula, if it is not in the scope of a quantifier $\forall x$ or $\exists x$.
- By $\varphi\frac{t}{x}$ we denote the result of replacing all free occurrences of the variable x in φ by the term t. (Bound variables are renamed.)

Semantics of formulas

$$[\![s = t]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![s]\!]^{\mathfrak{A}}_{\zeta} = [\![t]\!]^{\mathfrak{A}}_{\zeta}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\neg\varphi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{false}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\varphi \wedge \psi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true} \text{ and } [\![\psi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\varphi \vee \psi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true} \text{ or } [\![\psi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\varphi \to \psi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{false} \text{ or } [\![\psi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\forall x\, \varphi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta[x \mapsto a]} = \mathit{true} \text{ for every } a \in |\mathfrak{A}|; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

$$[\![\exists x\, \varphi]\!]^{\mathfrak{A}}_{\zeta} = \begin{cases} \mathit{true}, & \text{if there exists an } a \in |\mathfrak{A}| \text{ with } [\![\varphi]\!]^{\mathfrak{A}}_{\zeta[x \mapsto a]} = \mathit{true}; \\ \mathit{false}, & \text{otherwise.} \end{cases}$$

Coincidence, Substitution, Isomorphism

Lemma (Coincidence). If ζ and η are two variable assignments for φ such that $\zeta(x) = \eta(x)$ for all free variables x of φ, then $[\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = [\![\varphi]\!]^{\mathfrak{A}}_{\eta}$.

Lemma (Substitution). Let t be a term and $a = [\![t]\!]^{\mathfrak{A}}_{\zeta}$. Then $[\![\varphi\frac{t}{x}]\!]^{\mathfrak{A}}_{\zeta} = [\![\varphi]\!]^{\mathfrak{A}}_{\zeta[x\mapsto a]}$.

Lemma (Isomorphism). Let α be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Then $[\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = [\![\varphi]\!]^{\mathfrak{B}}_{\alpha\circ\zeta}$.

Models

Definition. A state $\mathfrak{A}$ is a *model* of φ (written $\mathfrak{A} \models \varphi$), if $[\![\varphi]\!]^{\mathfrak{A}}_{\zeta} = \mathit{true}$ for all variable assignments ζ for φ.

Part 3

Transition rules and runs of ASMs

Transition rules

Skip Rule: **skip**

Meaning: Do nothing

Update Rule: $f(s_1, \ldots, s_n) := t$

Meaning: Update the value of f at $(s_1, \ldots, s_n)$ to t.

Block Rule: P **par** Q

Meaning: P and Q are executed in parallel.

Conditional Rule: **if** φ **then** P **else** Q

Meaning: If φ is true, then execute P, otherwise execute Q.

Let Rule: **let** $x = t$ **in** P

Meaning: Assign the value of t to x and then execute P.

Transition rules (continued)

Forall Rule: $\boxed{\textbf{forall } x \textbf{ with } \varphi \textbf{ do } P}$

Meaning: Execute P in parallel for each x satisfying φ.

Choose Rule: $\boxed{\textbf{choose } x \textbf{ with } \varphi \textbf{ do } P}$

Meaning: Choose an x satisfying φ and then execute P.

Sequence Rule: $\boxed{P \textbf{ seq } Q}$

Meaning: P and Q are executed sequentially, first P and then Q.

Call Rule: $\boxed{r(t_1, \ldots, t_n)}$

Meaning: Call transition rule r with parameters $t_1, \ldots, t_n$.

Variations of the syntax

if φ **then**
P
else
Q
endif
	if φ **then** P **else** Q

[**do in-parallel**] P_1 $\vdots$ P_n [**enddo**]	P_1 **par** $\ldots$ **par** P_n
$\{P_1, \ldots, P_n\}$	P_1 **par** $\ldots$ **par** P_n

Variations of the syntax (continued)

do forall x: φ P **enddo**	**forall** x **with** φ **do** P
choose x: φ P **endchoose**	**choose** x **with** φ **do** P
step P **step** Q	P **seq** Q

Free and bound variables

Definition. An occurrence of a variable x is *free* in a transition rule, if it is not in the scope of a **let** x, **forall** x or **choose** x.

$$\textbf{let } x = t \underbrace{\textbf{in } P}_{\text{scope of } x}$$

$$\textbf{forall } x \underbrace{\textbf{with } \varphi \textbf{ do } P}_{\text{scope of } x}$$

$$\textbf{choose } x \underbrace{\textbf{with } \varphi \textbf{ do } P}_{\text{scope of } x}$$

Rule declarations

Definition. A *rule declaration* for a rule name r of arity n is an expression

$$\boxed{r(x_1, \ldots, x_n) = P}$$

where

- P is a transition rule and
- the free variables of P are contained in the list $x_1, \ldots, x_n$.

Remark: Recursive rule declarations are allowed.

Abstract State Machines

Definition. An *abstract state machine* M consists of

- a signature Σ,
- a set of initial states for Σ,
- a set of rule declarations,
- a distinguished rule name of arity zero called the *main rule name* of the machine.

Semantics of transition rules

The semantics of transition rules is defined in a calculus by rules:

$$\frac{\mathit{Premise}_1 \quad \cdots \quad \mathit{Premise}_n}{\mathit{Conclusion}} \; \mathit{Condition}$$

The predicate

$$\boxed{\mathsf{yields}(P, \mathfrak{A}, \zeta, U)}$$

means:

The transition rule P yields the update set U in state $\mathfrak{A}$ under the variable assignment ζ.

Semantics of transition rules (continued)

$$\frac{}{\mathsf{yields}(\textbf{skip}, \mathfrak{A}, \zeta, \emptyset)}$$

$$\frac{}{\mathsf{yields}(f(s_1, \ldots, s_n) := t, \mathfrak{A}, \zeta, \{(l, v)\})} \quad \text{where } l = (f, (\llbracket s_1 \rrbracket_\zeta^{\mathfrak{A}}, \ldots, \llbracket s_n \rrbracket_\zeta^{\mathfrak{A}})) \text{ and } v = \llbracket t \rrbracket_\zeta^{\mathfrak{A}}$$

$$\frac{\mathsf{yields}(P, \mathfrak{A}, \zeta, U) \quad \mathsf{yields}(Q, \mathfrak{A}, \zeta, V)}{\mathsf{yields}(P \textbf{ par } Q, \mathfrak{A}, \zeta, U \cup V)}$$

$$\frac{\mathsf{yields}(P, \mathfrak{A}, \zeta, U)}{\mathsf{yields}(\textbf{if } \varphi \textbf{ then } P \textbf{ else } Q, \mathfrak{A}, \zeta, U)} \quad \text{if } \llbracket \varphi \rrbracket_\zeta^{\mathfrak{A}} = \mathit{true}$$

$$\frac{\mathsf{yields}(Q, \mathfrak{A}, \zeta, V)}{\mathsf{yields}(\textbf{if } \varphi \textbf{ then } P \textbf{ else } Q, \mathfrak{A}, \zeta, V)} \quad \text{if } \llbracket \varphi \rrbracket_\zeta^{\mathfrak{A}} = \mathit{false}$$

$$\frac{\mathsf{yields}(P, \mathfrak{A}, \zeta[x \mapsto a], U)}{\mathsf{yields}(\textbf{let } x = t \textbf{ in } P, \mathfrak{A}, \zeta, U)} \quad \text{where } a = \llbracket t \rrbracket_\zeta^{\mathfrak{A}}$$

$$\frac{\mathsf{yields}(P, \mathfrak{A}, \zeta[x \mapsto a], U_a) \quad \text{for each } a \in I}{\mathsf{yields}(\textbf{forall } x \textbf{ with } \varphi \textbf{ do } P, \mathfrak{A}, \zeta, \bigcup_{a \in I} U_a)} \quad \text{where } I = \mathit{range}(x, \varphi, \mathfrak{A}, \zeta)$$

Semantics of transition rules (continued)

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta[x \mapsto a], U)}{\text{yields}(\textbf{choose } x \textbf{ with } \varphi \textbf{ do } P, \mathfrak{A}, \zeta, U)} \quad \text{if } a \in range(x, \varphi, \mathfrak{A}, \zeta)$$

$$\frac{}{\text{yields}(\textbf{choose } x \textbf{ with } \varphi \textbf{ do } P, \mathfrak{A}, \zeta, \emptyset)} \quad \text{if } range(x, \varphi, \mathfrak{A}, \zeta) = \emptyset$$

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta, U) \quad \text{yields}(Q, \mathfrak{A} + U, \zeta, V)}{\text{yields}(P \textbf{ seq } Q, \mathfrak{A}, \zeta, U \oplus V)} \quad \text{if } U \text{ is consistent}$$

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta, U)}{\text{yields}(P \textbf{ seq } Q, \mathfrak{A}, \zeta, U)} \quad \text{if } U \text{ is inconsistent}$$

$$\frac{\text{yields}(P\frac{t_1 \cdots t_n}{x_1 \cdots x_n}, \mathfrak{A}, \zeta, U)}{\text{yields}(r(t_1, \ldots, t_n), \mathfrak{A}, \zeta, U)} \quad \text{where } r(x_1, \ldots, x_n) = P \text{ is a rule declaration of } M$$

$$range(x, \varphi, \mathfrak{A}, \zeta) = \{a \in |\mathfrak{A}| : [\![\varphi]\!]^{\mathfrak{A}}_{\zeta[x \mapsto a]} = true\}$$

Coincidence, Substitution, Isomorphisms

Lemma (Coincidence). If $\zeta(x) = \eta(x)$ for all free variables x of a transition rule P and P yields U in $\mathfrak{A}$ under ζ, then P yields U in $\mathfrak{A}$ under η.

Lemma (Substitution). Let t be a static term and $a = [\![t]\!]^{\mathfrak{A}}_{\zeta}$. Then the rule $P\frac{t}{x}$ yields the update set U in state $\mathfrak{A}$ under ζ iff P yields U in $\mathfrak{A}$ under $\zeta[x \mapsto a]$.

Lemma (Isomorphism). If α is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and P yields U in $\mathfrak{A}$ under ζ, then P yields $\alpha(U)$ in $\mathfrak{B}$ under $\alpha \circ \zeta$.

Move of an ASM

Definition. A machine M can make a *move* from state $\mathfrak{A}$ to $\mathfrak{B}$ (written $\mathfrak{A} \stackrel{M}{\Longrightarrow} \mathfrak{B}$), if the main rule of M yields a consistent update set U in state $\mathfrak{A}$ and $\mathfrak{B} = \mathfrak{A} + U$.

- The updates in U are called *internal updates*.
- $\mathfrak{B}$ is called the *next internal state*.

If α is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A} & \stackrel{M}{\Longrightarrow} & \mathfrak{B} \\ \alpha \downarrow & & \downarrow \alpha \\ \mathfrak{A}' & \stackrel{M}{\Longrightarrow} & \mathfrak{B}' \end{array}$$

Run of an ASM

Let M be an ASM with signature Σ.

A *run* of M is a finite or infinite sequence $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ of states for Σ such that

- $\mathfrak{A}_0$ is an initial state of M
- for each n,
  - either M can make a move from $\mathfrak{A}_n$ into the next internal state $\mathfrak{A}'_n$ and the environment produces a consistent set of external or shared updates U such that $\mathfrak{A}_{n+1} = \mathfrak{A}'_n + U$,
  - or M cannot make a move in state $\mathfrak{A}_n$ and $\mathfrak{A}_n$ is the last state in the run.

- In *internal* runs, the environment makes no moves.
- In *interactive* runs, the environment produces updates.

Part 4

The reserve of ASMs

Importing new elements from the reserve

Import rule: $\boxed{\textbf{import } x \textbf{ do } P}$

Meaning: Choose an element x from the reserve, delete it from the reserve and execute P.

$\boxed{\textbf{let } x = new(X) \textbf{ in } P}$ abbreviates $\boxed{\begin{array}{l}\textbf{import } x \textbf{ do} \\ \quad X(x) := true \\ \quad P\end{array}}$

The reserve of a state

- New dynamic relation $Reserve$.
- $Reserve$ is updated by the system, not by rules.
- $Res(\mathfrak{A}) = \{a \in |\mathfrak{A}| : Reserve^{\mathfrak{A}}(a) = true\}$
- The reserve elements of a state are not allowed to be in the domain and range of any basic function of the state.

Definition. A state $\mathfrak{A}$ satisfies the *reserve condition* with respect to an environment ζ, if the following two conditions hold for each element $a \in Res(\mathfrak{A}) \setminus ran(\zeta)$:

- The element a is not the content of a location of $\mathfrak{A}$.
- If a is an element of a location l of $\mathfrak{A}$ which is not a location for $Reserve$, then the content of l in $\mathfrak{A}$ is $undef$.

Semantics of ASMs with a reserve

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta[x \mapsto a], U)}{\text{yields}(\textbf{import } x \textbf{ do } P, \mathfrak{A}, \zeta, V)} \quad \begin{array}{l}\text{if } a \in Res(\mathfrak{A}) \setminus ran(\zeta) \text{ and} \\ V = U \cup \{((Reserve, a), false)\}\end{array}$$

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta, U) \quad \text{yields}(Q, \mathfrak{A}, \zeta, V)}{\text{yields}(P \textbf{ par } Q, \mathfrak{A}, \zeta, U \cup V)} \quad \text{if } Res(\mathfrak{A}) \cap El(U) \cap El(V) \subseteq ran(\zeta)$$

$$\frac{\text{yields}(P, \mathfrak{A}, \zeta[x \mapsto a], U_a) \quad \text{for each } a \in I}{\text{yields}(\textbf{forall } x \textbf{ with } \varphi \textbf{ do } P, \mathfrak{A}, \zeta, \bigcup_{a \in I} U_a)} \quad \begin{array}{l}\text{if } I = range(x, \varphi, \mathfrak{A}, \zeta) \text{ and for } a \neq b \\ Res(\mathfrak{A}) \cap El(U_a) \cap El(U_b) \subseteq ran(\zeta)\end{array}$$

- $El(U)$ is the set of elements that occur in the updates of U.
- The elements of an update (l, v) are the value v and the elements of the location l.

Problem

Problem 1: New elements that are imported in parallel must be different.

import x **do** $parent(x) = root$
import y **do** $parent(y) = root$

Problem 2: Hiding of bound variables.

import x **do**
  $f(x) := 0$
  let $x = 1$ **in**
    import y **do** $f(y) := x$

Syntactic constraint. In the scope of a bound variable the same variable should not be used again as a bound variable (**let**, **forall**, **choose**, **import**).

Preservation of the reserve condition

Lemma (Preservation of the reserve condition).
If a state $\mathfrak{A}$ satisfies the reserve condition wrt. ζ and P yields a consistent update set U in $\mathfrak{A}$ under ζ, then

- the sequel $\mathfrak{A} + U$ satisfies the reserve condition wrt. ζ,
- $Res(\mathfrak{A} + U) \setminus ran(\zeta)$ is contained in $Res(\mathfrak{A}) \setminus El(U)$.

Permutation of the reserve

Lemma (Permutation of the reserve). Let $\mathfrak{A}$ be a state that satisfies the reserve condition wrt. ζ. If α is a function from $|\mathfrak{A}|$ to $|\mathfrak{A}|$ that permutes the elements in $Res(\mathfrak{A}) \setminus ran(\zeta)$ and is the identity on non-reserve elements of $\mathfrak{A}$ and on elements in the range of ζ, then α is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$.

Independence of the choice of reserve elements

Lemma (Independence).
Let P be a rule of an ASM without **choose**. If

- $\mathfrak{A}$ satisfies the reserve condition wrt. ζ,
- the bound variables of P are not in the domain of ζ,
- P yields U in $\mathfrak{A}$ under ζ,
- P yields U' in $\mathfrak{A}$ under ζ,

then there exists a permutation α of $Res(\mathfrak{A}) \setminus ran(\zeta)$ such that $\alpha(U) = U'$.