SS 2011 27. April 2011

Exercises for the Lecture Logics
Sheet 2

Prof. Dr. Klaus Madlener Delivery until 04. Mai 2011 10:00 Uhr

Exercise 1: [logical equivalence, satisfiability, tutorial]
Let p, q, r, and s be propositional variables and A, B, C formulas. Prove:

1. $\{p, p \vee q, p \to s, r \to q\} \models q \to p$
2. $\{p, p \vee q, p \to s, r \to q\} \models s$
3. $A \models\mathrel{\mkern-3mu}\Dashv \neg(\neg A))$
4. $A \wedge (B \wedge C) \models\mathrel{\mkern-3mu}\Dashv (A \wedge B) \wedge C$
5. $A \wedge (B \vee C) \models\mathrel{\mkern-3mu}\Dashv (A \wedge B) \vee (A \wedge C)$
6. $\neg(A \wedge B) \models\mathrel{\mkern-3mu}\Dashv (\neg A \vee \neg B)$ und $\neg(A \vee B) \models\mathrel{\mkern-3mu}\Dashv (\neg A \wedge \neg B)$
7. $A \to B \models\mathrel{\mkern-3mu}\Dashv (\neg A) \vee B$

Which different ways are there to prove these equivalences?

Exercise 2: [complete operator bases, tutorial]
Let the NAND-operator $|$ be defined by

$$\varphi(A \mid B) := \begin{cases} 0 & \text{if } \varphi(A) = \varphi(B) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Prove that $\{|\}$ is a complete operator basis.

Exercise 3: [boolean functions, tutorial]
Prove that every boolean function $f : \mathbb{B}^n \to \mathbb{B}$ can be represented by a formula built using $p_1, \ldots, p_n$ and an operator basis.

Exercise 4: [operator sets, tutorial]
Let $F(\{\neg, \leftrightarrow\})$ be the set of all formulas containing only variables, $\neg$, and $\leftrightarrow$.
Prove: If there are exactly n different variables in $A \in F(\{\neg, \leftrightarrow\})$, then there is a formula $A' \in F(\{\neg, \leftrightarrow\})$ with

1. $A \models\mathrel{\mkern-3mu}\Dashv A'$
2. Negation symbols occcur in A' only exactly in front of variables (e.g. $p \leftrightarrow \neg q$).
3. A' contains at most $2n$ Literals (i.e. occurrences of negated or non-negated variables).

Exercise 5: [semantic conclusion, 6P]
Prove or disprove:

1. $\{p \vee q, q \rightarrow r\} \models r$
2. $\{p \wedge q, \neg p \rightarrow (q \rightarrow r), q \wedge \neg r, \} \models q \rightarrow r$
3. $\{p, p \wedge q, p \rightarrow r, q \wedge \neg r, r \rightarrow s, (\neg q \vee r \vee \neg s) \rightarrow p\} \models (q \rightarrow r) \wedge (p \vee (r \rightarrow r))$
4. $\{p, p \rightarrow r, r \vee \neg q\} \models p \rightarrow (q \rightarrow p)$
5. $F \models q \rightarrow p \wedge (\neg(s \wedge \neg(s \vee ((q \wedge r) \rightarrow p))))$
6. $F \models p \leftrightarrow \neg p$

Exercise 6: [deduction theorem, 5P]

Prove the following variant of the deduction theorem:

$$\{A_1, \ldots, A_n\} \models B \text{ iff. } (A_1 \wedge \ldots \wedge A_n) \rightarrow B \text{ is a tautology.}$$

Exercise 7: [compactness theorem, 5P]

Prove that the following set is satisfiable:

$$\Sigma := \{p_i \vee p_{i+1} | i \in \mathbb{N}\} \cup \{(p_i \wedge p_{i+1}) \rightarrow \neg p_{i+2} | i \in \mathbb{N}\}$$

Exercise 8: [compactness theorem, 2P]

Let $\Sigma \subseteq F$ be an infinite set of propositional formulas and let $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$ be satisfiable subsets of F, s. th. $\Sigma' \subseteq \Sigma_i$ (for an $i > 0$) holds for every finite subset $\Sigma' \subseteq \Sigma$. Is Σ satisfiable? Prove your claim.

Exercise 9: [substitution, 4P]

Let $A, B, C \in F$ and let $A \models\!\!\!\dashv B$ be formulas and A be a subformula of C.

Prove: If C' is gained C by replacing one or more occurrences of A by B, then $C \models\!\!\!\dashv C'$ holds.

Exercise 10: [complete operator bases, 6P]

1. Prove that $\{\neg \rightarrow\}$ is a complete operator basis.
2. Prove that $\{\neg, \leftrightarrow\}$ *is not* a complete operator basis.

Exercise 11: [stapler, 1P]

Formalise the following proposition: If neither a stapler nor a paper clip is used for a submitted exercise, then possibly no points will be awarded for this exercise.

Attention: This proposition is a tautology!

Delivery: until 04. Mai 2011 10:00 Uhr into the box next to room 34-401.4