Exercises for the Lecture Logics
Sheet 3

Prof. Dr. Klaus Madlener   Delivery until 11. Mai 2011 10:00 Uhr

Exercise 1: [syntactical proofs for tautologies, tutorial]
  Consider the following rule patterns:

1. $\dfrac{A \vee \neg A}{true}$

2. $\dfrac{A \vee true}{true}$

3. $\dfrac{A \to \neg A}{\neg A}$

4. $\dfrac{(A \vee B) \vee C}{A \vee (B \vee C)}$

5. $\dfrac{A \vee B}{B \vee A}$

6. $\dfrac{A \to B}{\neg A \vee B}$

$true$ is a propositional constant with $\varphi(true) = 1$ for each valuation φ.

Show that the following propositions are tautologies by deducing $A \vdash true$ using the above rule patterns.

1. $A_1 \equiv (B \vee A) \vee (C \to \neg B)$

2. $A_2 \equiv p \to (q \to (p \to q))$

Exercise 2: [Proofs in deductive systems, tutorial]
  Prove:

1. $\neg(q \to p) \vdash_{\mathcal{F}_0} \neg p$

2. $\vdash_{\mathcal{F}_0} \neg(p \to p) \to \neg(p \to q)$

3. $(\neg(p \to q)) \vdash_{\mathcal{F}_0} (q \to p)$

Exercise 3: [Proofs in $\mathcal{F}_0$, 6P]
  Prove the formulas 10 and 11 from example 1.22 in the slides in $\mathcal{F}_0$.

Exercise 4: [Proofs by contradiction in $\mathcal{F}_0$, 6P]

1. Prove without using semantic arguments that $\Sigma \vdash_{\mathcal{F}_0} A$ holds if and only if $\Sigma \cup \{\neg A\}$ is inconsistent.

2. Prove in $\mathcal{F}_0$:

    a) $q, r \to \neg q \vdash_{\mathcal{F}_0} \neg r$

b) $A \rightarrow (\neg B \rightarrow C), (\neg B \rightarrow C) \rightarrow (A \rightarrow \neg B), \neg C \vdash_{\mathcal{F}_0} \neg A$

Exercise 5: [Correctness of Rules, 5P]

A rule pattern $R_0 : \dfrac{A_1, \ldots, A_n}{A}$ is *sound*, if, whenever the Premises $A_1, \ldots, A_n$ are tautologies, the conclusion A is also a tautology.

1. Prove: If $\mathcal{F} = (\text{Ax}, R)$ is a deductive system with sound axioms (i.e. the axioms are tautologies) and sound rules, then $\mathcal{F}$ is sound.

2. Find a sound rule $\dfrac{A_1, \ldots, A_n}{A}$ with $A_1, \ldots, A_n \not\models A$.

3. Find a simple deductive system with sound rules, s.t. every set of Formulas is inconsistent in the system.

Exercise 6: [Additional Operators in $\mathcal{F}_0$, 8P]

To cope with formulas from the whole set F in the deductive system $\mathcal{F}_0$ we can introduce further axioms.

1. Introduce further axioms in $\mathcal{F}_0$ which allow using formulas that contain the disjunctive operator ($\vee$). We will refer to this extended system as $\mathcal{F}_0'$.

2. Show that $\mathcal{F}_0'$ is still correct.

3. Find a formula Q from F_0 that is semantically equivalent to $P \equiv (p \vee q) \vee r$.

4. Prove without using semantical arguments that $\vdash_{\mathcal{F}_0'} P \rightarrow Q$ holds.

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