Chapter 2

Functional Programming:Isabelle

Overview of Isabelle/HOL

System Architecture

Isabelle	generic theorem prover

System Architecture

Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover

System Architecture

ProofGeneral	(X)Emacs based interface
Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover

HOL

HOL = Higher-Order Logic

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HOL has

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- recursive functions
- logical operators ($\wedge$, $\longrightarrow$, $\forall$, $\exists$, ...)

HOL is a programming language!

Higher-order = functions are values, too!

Formulae

Syntax (in decreasing priority):

$$
\begin{array}{lcllll}
form & ::= & (form) & | & term = term & | & \neg form \\
 & | & form \wedge form & | & form \vee form & | & form \longrightarrow form \\
 & | & \forall x.\ form & | & \exists x.\ form & &
\end{array}
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Examples

- $\neg\ A \wedge B \vee C \equiv ((\neg\ A) \wedge B) \vee C$

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- $\forall x.\ P\ x \wedge Q\ x \equiv \forall x.\ (P\ x \wedge Q\ x)$

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Scope of quantifiers: as far to the right as possible

Formulae

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  $A \wedge B \wedge C \equiv A \wedge (B \wedge C)$
- $A \longrightarrow B \longrightarrow C \equiv A \longrightarrow (B \longrightarrow C) \not\equiv (A \longrightarrow B) \longrightarrow C$!

Warning

Quantifiers have low priority and need to be parenthesized:

! $P \wedge \forall x.\ Q\ x \rightsquigarrow P \wedge (\forall x.\ Q\ x)$!

Types and Terms

Types

Syntax:

$$\tau \quad ::= \quad (\tau)$$
$$\quad\quad | \quad \mathit{bool} \mid \mathit{nat} \mid \ldots \qquad \text{base types}$$

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Parentheses: *T1* $\Rightarrow$ *T2* $\Rightarrow$ *T3* $\equiv$ *T1* $\Rightarrow$ *(T2* $\Rightarrow$ *T3)*

Terms: Basic syntax

Syntax:

$term$	$::=$	$(term)$	
	$\mid$	a	constant or variable (identifier)
	$\mid$	$term\ term$	function application
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Parantheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$

λ-calculus on one slide

Informal notation: $t[x]$

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  Replace formal by actual parameter ("β-reduction"):
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Example: $(\lambda\ x.\ x + 5)\ 3 \quad \longrightarrow_\beta \quad (3 + 5)$

$\longrightarrow_\beta$ *in Isabelle: Don't worry, be happy*

Isabelle performs β-reduction automatically

Isabelle considers $(\lambda x.t[x])a$ and $t[a]$ equivalent

Terms and Types

Terms must be well-typed

(the argument of every function call must be of the right type)

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Notation: $t :: \tau$ means t is a well-typed term of type τ.

Type inference

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In the presence of *overloaded* functions (functions with multiple types) not always possible.

User can help with type annotations inside the term.

Example: *f (x::nat)*

Currying

Thou shalt curry your functions

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- Curried: *f* :: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: *f'* :: $\tau_1 \times \tau_2 \Rightarrow \tau$

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Advantage: *partial application* *f* a_1 with a_1 :: τ_1

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix:* +, -, *, #, @, . . .
- *Mixfix:* *if _ then _ else _*, *case _ of*, . . .

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Prefix binds more strongly than infix:

! $f\ x + y \equiv (f\ x) + y \not\equiv f\ (x + y)$!

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix:* +, -, *, #, @, ...
- *Mixfix:* *if _ then _ else _*, *case _ of*, ...

Prefix binds more strongly than infix:

! $f\ x + y \equiv (f\ x) + y \not\equiv f\ (x + y)$!

Enclose *if* and *case* in parentheses:

! *(if _ then _ else _)* !

Base types: bool, nat, list

Type bool

Formulae = terms of type *bool*

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True :: bool
False :: bool
$\wedge, \vee, \ldots$ *:: bool* $\Rightarrow$ *bool* $\Rightarrow$ *bool*
$\vdots$

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if-and-only-if: *=*

Type nat

$0 :: nat$
$Suc :: nat \Rightarrow nat$
$+, *, ... :: nat \Rightarrow nat \Rightarrow nat$
$\vdots$

Type nat

0 :: nat
Suc :: nat ⇒ nat
*+, *, ... :: nat ⇒ nat ⇒ nat*
⋮

! Numbers and arithmetic operations are overloaded:
0,1,2,... :: 'a, + :: 'a ⇒ 'a ⇒ 'a

You need type annotations: *1 :: nat, x + (y::nat)*

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0,1,2,... :: 'a, + :: 'a ⇒ 'a ⇒ 'a

You need type annotations: *1 :: nat, x + (y::nat)*

. . . unless the context is unambiguous: *Suc z*

Type list

- *[]*: empty list
- *x # xs*: list with first element *x* ("*head*") and rest *xs* ("*tail*")
- Syntactic sugar: $[x_1,\ldots,x_n]$

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Large library:
hd, *tl*, *map*, *length*, *filter*, *set*, *nth*, *take*, *drop*, *distinct*, ...

Don't reinvent, reuse!
$\leadsto$ `HOL/List.thy`

Isabelle Theories

Theory = Module

Syntax: `theory` $MyTh$
`imports` $ImpTh_1 \ldots ImpTh_n$
`begin`
(declarations, definitions, theorems, proofs, ...)*
`end`

- $MyTh$: name of theory. Must live in file $MyTh$`.thy`
- $ImpTh_i$: name of *imported* theories. Import transitive.

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- $MyTh$: name of theory. Must live in file $MyTh$`.thy`
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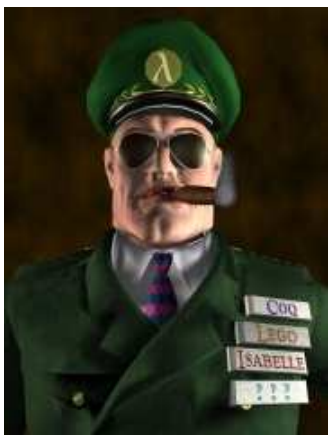
Usually:

```
theory MyTh
imports Main
⋮
```

Proof General



An Isabelle Interface

by David Aspinall

Proof General

Customized version of (x)emacs:

- all of emacs (info: `C-h i`)
- Isabelle aware (when editing `.thy` files)
- mathematical symbols (“x-symbols”)

X-Symbols

Input of funny symbols in Proof General

- via menu (“X-Symbol”)
- via ascii encoding (similar to LaTeX): `\<and>`, `\<or>`, ...
- via abbreviation: `/\`, `\/`, `-->`, ...

x-symbol	∀	∃	λ	¬	∧	∨	⟶	⇒
ascii (1)	`\<forall>`	`\<exists>`	`\<lambda>`	`\<not>`	`/\`	`\/`	`-->`	`=>`
ascii (2)	`ALL`	`EX`	`%`	`~`	`&`	|		

(1) is converted to x-symbol, (2) stays ascii.

Demo: terms and types

An introduction to recursion and induction

A recursive datatype: toy lists

datatype *'a list = Nil | Cons 'a ('a list)*

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***Nil*:** empty list

***Cons x xs*:** head *x :: 'a*, tail *xs :: 'a list*

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A toy list: *Cons False (Cons True Nil)*

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***Nil*:** empty list

***Cons x xs*:** head *x :: 'a*, tail *xs :: 'a list*

A toy list: *Cons False (Cons True Nil)*

Predefined lists: *[False, True]*

Structural induction on lists

P xs holds for all lists *xs* if

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- *P Nil*

Structural induction on lists

P xs holds for all lists *xs* if

- *P Nil*
- and for arbitrary *x* and *xs*, *P xs* implies *P (Cons x xs)*

A recursive function: append

Definition by *primitive recursion*:

primrec *app :: 'a list ⇒ 'a list ⇒ 'a list* **where**
app Nil ys = ? |
app (Cons x xs) ys = ??

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1 rule per constructor
Recursive calls must drop the constructor $\Longrightarrow$ Termination

Concrete syntax

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Except for single identifiers, e.g. *'a*

" normally not shown on slides

Demo: append and reverse

Proofs

General schema:

lemma *name*`: "..."`
apply `(...)`
apply `(...)`
⋮
done

If the lemma is suitable as a simplification rule:

lemma *name*`[simp]: "..."`

Proof methods

- Structural induction
  - Format: *(induct x)*
    x must be a free variable in the first subgoal.
    The type of *x* must be a datatype.
  - Effect: generates 1 new subgoal per constructor
- Simplification and a bit of logic
  - Format: *auto*
  - Effect: tries to solve as many subgoals as possible using simplification and basic logical reasoning.

Top down proofs

Command

sorry

"completes" any proof.

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Allows top down development:

Assume lemma first, prove it later.

Some useful tools

Disproving tools

Automatic counterexample search by random testing:
quickcheck

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Automatic counterexample search by random testing:

quickcheck

Counterexample search via SAT solver:

nitpick

Finding theorems

1. Click on Find button
2. Input search pattern (e.g. *"_ & True"*)

Demo: Disproving and Finding

Isabelle's meta-logic

Basic constructs

Implication $\Longrightarrow$ (==>)

For separating premises and conclusion of theorems

Basic constructs

Implication $\Longrightarrow$ (**==>**)

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Equality $\equiv$ (**==**)

For definitions

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Universal quantifier $\bigwedge$ (**! !**)
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Do not use *inside* HOL formulae

Notation

$$[\![A_1; \ldots ; A_n]\!] \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \ldots \Longrightarrow A_n \Longrightarrow B$$

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$;$ $\approx$ "and"

The proof state

$$1.\ \bigwedge x_1 \ldots x_p.\ [\![A_1; \ldots ; A_n]\!] \Longrightarrow B$$

$x_1 \ldots x_p$	Local constants
$A_1 \ldots A_n$	Local assumptions
B	Actual (sub)goal

Type and function definition in Isabelle/HOL

Type definition in Isabelle/HOL

Introducing new types

Keywords:

- **typedecl**: pure declaration
- **types**: abbreviation
- **datatype**: recursive datatype

typedecl

typedecl $name$

Introduces new “opaque” type $name$ without definition

typedecl

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Example:

typedecl *addr* — An abstract type of addresses

types

types $name = \tau$

Introduces an *abbreviation* $name$ for type τ

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Examples:

types
name = string
('a,'b)foo = 'a list $\times$ *'b list*

types

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Type abbreviations are expanded immediately after parsing
Not present in internal representation and Isabelle output

datatype

The example

datatype *'a list = Nil | Cons 'a ('a list)*

Properties:

- Types: *Nil :: 'a list*
  Cons :: 'a ⇒ 'a list ⇒ 'a list
- Distinctness: *Nil ≠ Cons x xs*
- Injectivity: *(Cons x xs = Cons y ys) = (x = y ∧ xs = ys)*

The general case

datatype $(\alpha_1, \ldots, \alpha_n)\tau$ = $C_1\ \tau_{1,1} \ldots \tau_{1,n_1}$
| $\ldots$
| $C_k\ \tau_{k,1} \ldots \tau_{k,n_k}$

- *Types:* $C_i :: \tau_{i,1} \Rightarrow \cdots \Rightarrow \tau_{i,n_i} \Rightarrow (\alpha_1, \ldots, \alpha_n)\tau$
- *Distinctness:* $C_i\ \ldots \neq C_j\ \ldots$ if $i \neq j$
- *Injectivity:*
  $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

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  $(C_i\ x_1 \ldots x_{n_i} = C_i\ y_1 \ldots y_{n_i}) = (x_1 = y_1 \wedge \ldots \wedge x_{n_i} = y_{n_i})$

Distinctness and Injectivity are applied automatically
Induction must be applied explicitly

Function definition in Isabelle/HOL

Why nontermination can be harmful

How about $f\ x = f\ x + 1$?

Why nontermination can be harmful

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Subtract $f\ x$ on both sides.

$$\Longrightarrow 0 = 1$$

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! All functions in HOL must be total !

Function definition schemas in Isabelle/HOL

- Non-recursive with **definition**
  No problem

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  Terminating by construction
- Well-founded recursion with **fun**
  Automatic termination proof
- Well-founded recursion with **function**
  User-supplied termination proof

definition

Definition (non-recursive) by example

definition *sq :: nat ⇒ nat* **where** *sq n = n * n*

Definitions: pitfalls

definition *prime :: nat ⇒ bool* **where**
prime p = (1 < p ∧ (m dvd p ⟶ m = 1 ∨ m = p))

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Not a definition: free *m* not on left-hand side

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prime p = (1 < p ∧ (∀ m. m dvd p ⟶ m = 1 ∨ m = p))

Using definitions

Definitions are not used automatically

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Unfolding the definition of *sq*:

apply*(unfold sq_def)*

primrec

The example

primrec *app :: 'a list ⇒ 'a list ⇒ 'a list* **where**

app Nil ys = ys

app (Cons x xs) ys = Cons x (app xs ys)

The general case

If τ is a datatype (with constructors $C_1, \ldots, C_k$) then $f :: \cdots \Rightarrow \tau \Rightarrow \cdots \Rightarrow \tau'$ can be defined by *primitive recursion*:

$$
\begin{array}{lcl}
f\ x_1 \ldots (C_1\ y_{1,1} \ldots y_{1,n_1}) \ldots x_p & = & r_1 \mid \\
\vdots & & \\
f\ x_1 \ldots (C_k\ y_{k,1} \ldots y_{k,n_k}) \ldots x_p & = & r_k
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The recursive calls in r_i must be *structurally smaller*, i.e. of the form $f\ a_1 \ldots y_{i,j} \ldots a_p$

nat is a datatype

datatype *nat = 0 | Suc nat*

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Functions on *nat* definable by primrec!

primrec *f :: nat ⇒ ...*
f 0 = ...
f(Suc n) = ... f n ...

More predefined types and functions

Type option

datatype *'a option = None | Some 'a*

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Important application:

$$\ldots \Rightarrow \textit{'a option} \approx \text{partial function:}$$

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Example:

primrec *lookup :: 'k ⇒ ('k × 'v) list ⇒ 'v option* **where**
lookup k [] = None |
lookup k (x#xs) =
(if fst x = k then Some(snd x) else lookup k xs)

case

Datatype values can be taken apart with *case* expressions:

(case xs of [] ⇒ . . . | y#ys ⇒ ... y ... ys ...)

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(case xs of [0] ⇒ 0 | [Suc n] ⇒ n | _ ⇒ 2)

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Needs *( )* in context

Proof by case distinction

If $t :: \tau$ and τ is a datatype

apply(*case_tac* t)

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If $t :: \tau$ and τ is a datatype

apply(*case_tac* t)

creates k subgoals

$$t = C_i\ x_1 \ldots x_p \Longrightarrow \ldots$$

one for each constructor C_i of type τ.

Demo: trees

fun

From primitive recursion to arbitrary pattern matching

Example: Fibonacchi

fun *fib :: nat* $\Rightarrow$ *nat* **where**

fib 0 = 0 |
fib (Suc 0) = 1 |
fib (Suc(Suc n)) = fib (n+1) + fib n

Example: Separation

fun *sep :: 'a ⇒ 'a list ⇒ 'a list* **where**

sep a [] = [] |
sep a [x] = [x] |
sep a (x#y#zs) = x # a # sep a (y#zs)

Example: Ackermann

fun *ack :: nat ⇒ nat ⇒ nat* **where**

```
ack 0       n       = Suc n  |
ack (Suc m) 0       = ack m (Suc 0)  |
ack (Suc m) (Suc n) = ack m (ack (Suc m) n)
```

Key features of fun

- Arbitrary pattern matching

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- Termination must be provable
  by lexicographic combination of size measures

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- *size* counts number of (non-nullary) constructors

Lexicographic ordering

Either the first component decreases, or it stays unchanged and the second component decreases:

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Theorem If each component ordering terminates, then their *lexicographic product* terminates, too.

Ackermann terminates

ack 0 n = Suc n

ack (Suc m) 0 = ack m (Suc 0)

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Note: order of arguments not important for Isabelle!

Computation Induction

If $f :: \tau \Rightarrow \tau'$ is defined by **fun**, a special induction schema is provided to prove $P(x)$ for all $x :: \tau$:

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Induction follows course of (terminating!) computation

Computation Induction: Example

fun *div2 :: nat ⇒ nat* **where**
div2 0 = 0 |
div2 (Suc 0) = 0 |
div2(Suc(Suc n)) = Suc(div2 n)

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⇝ induction rule `div2.induct`:

$$\frac{P(0) \quad P(Suc\ 0) \quad P(n) \Longrightarrow P(Suc(Suc\ n))}{P(m)}$$

Demo: fun