Gentzen Calculus

Proof Systems / Logical Calculi

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Gentzen Calculus

Consists of:

Gentzen Calculus

Consists of:

One **Axiom** $A \vdash A$

Gentzen Calculus

Consists of:

One **Axiom** $A \vdash A$

Multiple **Rules**

Gentzen Calculus

Consists of:

One **Axiom** $A \vdash A$

Multiple **Rules**

Weakening (W) $$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

Gentzen Calculus

Consists of:

One **Axiom** $\qquad A \vdash A$

Multiple **Rules**

Weakening (W)
$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

Conjunction (^I)
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

Conjunction (^E$_I$)
$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

Conjunction (^E$_R$)
$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

Gentzen Calculus

Multiple **Rules**

Disjunction (∨I_l)

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

Disjunction (∨I_r)

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Disjunction (∨E)

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Gentzen Calculus

Multiple **Rules**

Disjunction (∨I$_l$)
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

Disjunction (∨I$_r$)
$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Disjunction (∨E)
$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Truth Values (FalseE)
$$\frac{\Gamma \vdash \textbf{False}}{\Gamma \vdash A}$$

Gentzen Calculus

Multiple **Rules**

Disjunction (∨I$_l$)
$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

Disjunction (∨I$_r$)
$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Disjunction (∨E)
$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Truth Values (FalseE)
$$\frac{\Gamma \vdash \text{False}}{\Gamma \vdash A}$$

Negation (¬I)
$$\frac{\Gamma, A \vdash \text{False}}{\Gamma \vdash \neg A}$$

Negation (¬E)
$$\frac{\Gamma \vdash \neg A \qquad \Gamma \vdash A}{\Gamma \vdash \text{False}}$$

Gentzen Calculus

Multiple **Rules**

Implication (→I)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$

Implication (→E)
Modus Ponens

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Gentzen Calculus

Multiple **Rules**

Implication ($\rightarrow$I)
$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

Implication ($\rightarrow$E)
Modus Ponens
$$\frac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Until here: *Ready for Exercise 1.a*

Gentzen Calculus

Exercise 1.a - Warmup

$$\overline{\{A \rightarrow C, B \rightarrow C\} \vdash (A \vee B) \rightarrow C}$$

Gentzen Calculus

Exercise 1.a - Warmup

$$\frac{\{A \to C, B \to C, A \vee B\} \vdash C}{\{A \to C, B \to C\} \vdash (A \vee B) \to C} \boxed{\to I}$$

Gentzen Calculus

Exercise 1.a - Warmup

possible choices

Conjunction (^EI) $$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$ Conjunction (^ER) $$\dfrac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

Implication ($\rightarrow$E)
Modus Ponens $$\dfrac{\Gamma \vdash A \rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

Disjunction (vE) $$\dfrac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

$$?$$

$$\dfrac{\{A \rightarrow C, B \rightarrow C, A \vee B\} \vdash C}{\{A \rightarrow C, B \rightarrow C\} \vdash (A \vee B) \rightarrow C} \quad \boxed{\rightarrow\text{I}}$$

Gentzen Calculus

Exercise 1.a - Warmup

$\boxed{\vee E}$ $\quad \dfrac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$ $\qquad$ *Axiom:* $A \vdash A$

Axiom

$$\dfrac{\dfrac{\quad}{\Gamma \vdash A \vee B} \qquad \dfrac{\quad}{\Gamma, A \vdash C} \qquad \dfrac{\quad}{\Gamma, B \vdash C}}{\dfrac{\Gamma := \{A \to C, B \to C, A \vee B\} \vdash C}{\{A \to C, B \to C\} \vdash (A \vee B) \to C} \;\boxed{\to I}} \;\boxed{\vee E}$$

Gentzen Calculus

Exercise 1.a - Warmup

$$\boxed{\text{vE}} \quad \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C} \qquad \textit{Axiom:} \; A \vdash A$$

$$\cfrac{\textit{Axiom} \quad \cfrac{\cfrac{\textit{Axiom}}{\Gamma, A \vdash A \to C} \quad \cfrac{\textit{Axiom}}{\Gamma, A \vdash A}}{\Gamma, A \vdash C} \boxed{\to E} \quad \cfrac{\cfrac{\textit{Axiom}}{\Gamma, B \vdash A \to C} \quad \cfrac{\textit{Axiom}}{\Gamma, B \vdash A}}{\Gamma, B \vdash C} \boxed{\to E}}{\cfrac{\Gamma := \{A \to C, B \to C, A \vee B\} \vdash C}{\{A \to C, B \to C\} \vdash (A \vee B) \to C} \boxed{\to I}} \boxed{\text{vE}}$$

With $\cfrac{\textit{Axiom}}{\Gamma \vdash A \vee B}$

Gentzen Calculus

Exercise 1.a - Warmup

vE

$$\frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

Axiom: $A \vdash A$

! Proven !

$$\frac{\cfrac{Axiom}{\Gamma \vdash A \vee B} \qquad \cfrac{\cfrac{Axiom}{\Gamma, A \vdash A \rightarrow C} \quad \cfrac{Axiom}{\Gamma, A \vdash A}}{\Gamma, A \vdash C} \;\rightarrow E \qquad \cfrac{\cfrac{Axiom}{\Gamma, B \vdash A \rightarrow C} \quad \cfrac{Axiom}{\Gamma, B \vdash A}}{\Gamma, B \vdash C}\;\rightarrow E}{\Gamma := \{A \rightarrow C, B \rightarrow C, A \vee B\} \vdash C}\;vE$$

$$\frac{\Gamma := \{A \rightarrow C, B \rightarrow C, A \vee B\} \vdash C}{\{A \rightarrow C, B \rightarrow C\} \vdash (A \vee B) \rightarrow C}\;\rightarrow I$$

Gentzen Calculus

Exercise 1.a

$$\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))$$

Gentzen Calculus

Exercise 1.a

$$\frac{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \; \boxed{\rightarrow I}$$

Gentzen Calculus

Exercise 1.a

$$a \vee (b \wedge c) \vdash a \vee b \qquad \textit{Left side} \qquad\qquad\qquad a \vee (b \wedge c) \vdash a \vee c$$

$$\frac{\qquad\qquad a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c) \qquad}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \; \boxed{\rightarrow\!\text{I}}$$

$\boxed{\wedge\text{I}}$

Gentzen Calculus

Exercise 1.a

$$\frac{}{a \vee (b \wedge c) \vdash a \vee (b \wedge c)} \qquad \frac{}{a \vee (b \wedge c), a \vdash a \vee b} \qquad \frac{}{a \vee (b \wedge c), (b \wedge c) \vdash a \vee b}$$

VE

$$\boxed{a \vee (b \wedge c) \vdash a \vee b} \quad \textit{Left side} \qquad\qquad a \vee (b \wedge c) \vdash a \vee c$$

∧I

$$a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)$$

→I

$$\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))$$

Gentzen Calculus

Exercise 1.a

Axiom

Axiom

$$\frac{}{a \vee (b \wedge c) \vdash a \vee (b \wedge c)}$$

$$\frac{\overline{a \vdash a \vee b}}{a \vee (b \wedge c), a \vdash a \vee b} \boxed{W}$$

$$\frac{\overline{b \wedge c \vdash a \vee b}}{a \vee (b \wedge c), (b \wedge c) \vdash a \vee b} \boxed{W}$$

$$\boxed{VE}$$

$$\boxed{a \vee (b \wedge c) \vdash a \vee b} \quad \textit{Left side}$$

$$a \vee (b \wedge c) \vdash a \vee c$$

$$\boxed{\wedge I}$$

$$\frac{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \boxed{\rightarrow I}$$

Gentzen Calculus

Exercise 1.a

Axiom

$$\dfrac{}{a \vdash a}$$

$$\dfrac{a \vdash a}{a \vdash a \vee b} \;\text{∨I}_I$$

$$\dfrac{a \vdash a \vee b}{a \vee (b \wedge c), a \vdash a \vee b} \;\text{W}$$

$$\dfrac{}{b \wedge c \vdash b}$$

$$\dfrac{b \wedge c \vdash b}{b \wedge c \vdash a \vee b} \;\text{∨I}_R$$

$$\dfrac{b \wedge c \vdash a \vee b}{a \vee (b \wedge c), (b \wedge c) \vdash a \vee b} \;\text{W}$$

Axiom

$$\dfrac{}{a \vee (b \wedge c) \vdash a \vee (b \wedge c)}$$

$$\text{VE}$$

$$\boxed{a \vee (b \wedge c) \vdash a \vee b} \quad \textit{Left side}$$

$$a \vee (b \wedge c) \vdash a \vee c$$

$$\dfrac{a \vee (b \wedge c) \vdash a \vee b \qquad a \vee (b \wedge c) \vdash a \vee c}{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)} \;\text{∧I}$$

$$\dfrac{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \to ((a \vee b) \wedge (a \vee c))} \;\to\!\text{I}$$

Gentzen Calculus

Exercise 1.a

Axiom

Axiom

$$\dfrac{b \wedge c \;\vdash\; b \wedge c}{b \wedge c \;\vdash\; b} \quad \wedge E_l$$

$$\dfrac{a \vdash a}{a \vdash a \vee b} \quad \vee I_l$$

$$\dfrac{b \wedge c \;\vdash\; a \vee b}{} \quad \vee I_R$$

$$\dfrac{}{a \vdash a \vee b} \quad W$$

Axiom

$$\dfrac{}{a \vee (b \wedge c) \vdash a \vee (b \wedge c)}$$

$$\dfrac{}{a \vee (b \wedge c), a \vdash a \vee b} \quad W \qquad \dfrac{}{a \vee (b \wedge c), (b \wedge c) \vdash a \vee b} \quad W$$

$$\dfrac{}{\boxed{a \vee (b \wedge c) \;\vdash\; a \vee b}} \quad \text{\textit{Left side}} \qquad \qquad \quad \text{VE}$$

$$a \vee (b \wedge c) \;\vdash\; a \vee c$$

$$\dfrac{}{a \vee (b \wedge c) \;\vdash\; (a \vee b) \wedge (a \vee c)} \quad \wedge I$$

$$\dfrac{}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \quad \rightarrow I$$

Gentzen Calculus

Exercise 1.a

$$a \vee (b \wedge c) \vdash a \vee b \qquad \qquad \textit{Right side} \quad \boxed{a \vee (b \wedge c) \vdash a \vee c} \quad \boxed{\wedge\text{I}}$$

$$\dfrac{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \quad \boxed{\rightarrow\text{I}}$$

Gentzen Calculus

Exercise 1.a

$$\frac{\overset{\textit{Axiom}}{\overline{a \vee (b \wedge c) \vdash a \vee (b \wedge c)}}}{a \vee (b \wedge c) \vdash a \vee b}$$

$$\overline{a \vee (b \wedge c), a \vdash a \vee c} \qquad \overline{a \vee (b \wedge c), (b \wedge c) \vdash a \vee c}$$

$$\textit{Right side} \quad \boxed{a \vee (b \wedge c) \vdash a \vee c} \quad \text{VE}$$

$$\frac{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \to ((a \vee b) \wedge (a \vee c))} \quad \to\! I$$

$\wedge I$

Gentzen Calculus

Exercise 1.a

Axiom

$$\frac{}{a \vee (b \wedge c) \vdash a \vee (b \wedge c)}$$

$$\frac{\dfrac{}{a \vdash a \vee c}}{a \vee (b \wedge c), a \vdash a \vee c} \quad \boxed{W}$$

$$\frac{\dfrac{}{b \wedge c \vdash a \vee c}}{a \vee (b \wedge c), (b \wedge c) \vdash a \vee c} \quad \boxed{W}$$

$$a \vee (b \wedge c) \vdash a \vee b$$

Right side $\boxed{a \vee (b \wedge c) \vdash a \vee c}$

$$\frac{}{a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)} \quad \boxed{\wedge I}$$

$$\frac{}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \quad \boxed{\rightarrow I}$$

Gentzen Calculus

Exercise 1.a

Axiom

$$\dfrac{\overline{a \vdash a} \quad \text{vI}_\text{I}}{a \vdash a \lor c} \quad \text{W} \atop a \lor (b \land c), a \vdash a \lor c$$

$$\dfrac{\overline{b \land c \ \vdash \ c} \quad \text{vI}_\text{R}}{b \land c \ \vdash a \lor c} \quad \text{W} \atop a \lor (b \land c), (b \land c) \vdash a \lor c$$

Axiom

$$\overline{a \lor (b \land c) \vdash a \lor (b \land c)}$$

$$a \lor (b \land c) \vdash \ a \lor b$$

Right side $\boxed{a \lor (b \land c) \ \vdash \ a \lor c}$

$$\dfrac{a \lor (b \land c) \vdash (a \lor b) \land (a \lor c)}{\vdash (a \lor (b \land c)) \to ((a \lor b) \land (a \lor c))} \quad \to\text{I}$$

$\land\text{I}$

Gentzen Calculus

Exercise 1.a

$$\text{Axiom} \quad \frac{}{b \wedge c \ \vdash \ b \wedge c} \quad \wedge E_R$$

$$\frac{b \wedge c \ \vdash \ c}{\ } \quad \vee I_R$$

$$\text{Axiom} \quad \frac{}{a \vdash a} \quad \vee I_I$$

$$\frac{a \vdash a \vee c}{\ } \quad W$$

$$\frac{b \wedge c \ \vdash \ a \vee c}{\ } \quad W$$

$$\text{Axiom} \quad \frac{}{a \vee (b \wedge c) \ \vdash \ a \vee (b \wedge c)}$$

$$a \vee (b \wedge c),\, a \vdash a \vee c$$

$$a \vee (b \wedge c),\, (b \wedge c) \vdash a \vee c$$

$$a \vee (b \wedge c) \ \vdash \ a \vee b \qquad \text{Right side} \quad \boxed{a \vee (b \wedge c) \ \vdash \ a \vee c} \quad \wedge I$$

$$\frac{a \vee (b \wedge c) \ \vdash \ (a \vee b) \wedge (a \vee c)}{\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))} \quad \rightarrow I$$

Gentzen Calculus

Exercise 1.a

All Axioms

All Axioms

$$a \vee (b \wedge c) \vdash a \vee b$$

$$a \vee (b \wedge c) \vdash a \vee c$$

$\wedge$I

$$a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)$$

$\rightarrow$I

$$\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))$$

Gentzen Calculus

Exercise 1.a

! Proven !

All Axioms

All Axioms

$$a \vee (b \wedge c) \vdash a \vee b$$

$$a \vee (b \wedge c) \vdash a \vee c$$

∧I

$$a \vee (b \wedge c) \vdash (a \vee b) \wedge (a \vee c)$$

→I

$$\vdash (a \vee (b \wedge c)) \rightarrow ((a \vee b) \wedge (a \vee c))$$

Gentzen Calculus

Exercise 1.b

Additional **Rules**

All-quantor

$$(\forall I) \quad \frac{\Gamma \vdash \{a_{new}/x\}A}{\Gamma \vdash \forall x.A} \qquad\qquad (\forall E) \quad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash \{t/x\}A}$$

Existence-quantor

$$(\exists I) \quad \frac{\Gamma \vdash \{t/x\}A}{\Gamma \vdash \exists x.A} \qquad (\exists E) \quad \frac{\Gamma \vdash \exists x.A \qquad \Gamma, \{a_{new}/x\}A \vdash C}{\Gamma \vdash C}$$

Gentzen Calculus

Exercise 1.b

Additional **Rules**

All-quantor

$$(\forall I) \quad \frac{\Gamma \vdash \{a_{new}/x\}A}{\Gamma \vdash \forall x.A} \qquad\qquad (\forall E) \quad \frac{\Gamma \vdash \forall x.A}{\Gamma \vdash \{t/x\}A}$$

Existence-quantor

$$(\exists I) \quad \frac{\Gamma \vdash \{t/x\}A}{\Gamma \vdash \exists x.A} \qquad (\exists E) \quad \frac{\Gamma \vdash \exists x.A \qquad \Gamma, \{a_{new}/x\}A \vdash C}{\Gamma \vdash C}$$

Until here: *Ready for Exercise 1.b*

Gentzen Calculus

Exercise 1.b

$$\vdash \exists x. \forall y. P(x, y) \rightarrow \forall y. \exists x. P(x, y)$$

Gentzen Calculus

Exercise 1.b

$$\frac{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}{\vdash \exists x.\forall y.P(x,y) \rightarrow \forall y.\exists x.P(x,y)} \quad \boxed{\rightarrow \text{I}}$$

Gentzen Calculus

Exercise 1.b   **! Proven !**

$$\cfrac{\cfrac{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)}{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)} \; \forall I}{\vdash \exists x.\forall y.P(x,y) \rightarrow \forall y.\exists x.P(x,y)} \; \rightarrow I$$

Gentzen Calculus

Exercise 1.b   **! Proven !**

Axiom

$$\overline{\exists x.\forall y.P(x,y) \vdash \exists x.\forall y.P(x,y)} \qquad \overline{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)}$$

$$\cfrac{\cfrac{}{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)} \; \exists E}{\cfrac{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}{\vdash \exists x.\forall y.P(x,y) \to \forall y.\exists x.P(x,y)} \; \to I} \; \forall I$$

Gentzen Calculus

Exercise 1.b   **! Proven !**

$$\dfrac{\dfrac{\exists x.\forall y.P(x,y) \vdash \exists x.\forall y.P(x,y) \qquad \dfrac{\dfrac{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash P(b,a)}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)}\; \boxed{\exists I}}{}}{\dfrac{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)}{\dfrac{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}{\vdash \exists x.\forall y.P(x,y) \rightarrow \forall y.\exists x.P(x,y)}\; \boxed{\rightarrow I}}\; \boxed{\forall I}}\; \boxed{\exists E}}{}$$

Axiom

Gentzen Calculus

Exercise 1.b   **! Proven !**

$$\cfrac{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \forall y.P(b,y)}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash P(b,a)} \; \forall E$$

Axiom

$$\cfrac{\exists x.\forall y.P(x,y) \vdash \exists x.\forall y.P(x,y) \qquad \cfrac{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)}{\qquad} \; \exists I}{\cfrac{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)}{\cfrac{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}{\vdash \exists x.\forall y.P(x,y) \to \forall y.\exists x.P(x,y)} \; {\to} I} \; \forall I} \; \exists E$$

Gentzen Calculus

Exercise 1.b

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{Axiom}}{\forall y.P(b,y) \vdash \forall y.P(b,y)}}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \forall y.P(b,y)} \; W}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash P(b,a)} \; \forall E}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)} \; \exists I}{\qquad}$$

$$\cfrac{\cfrac{\text{Axiom}}{\exists x.\forall y.P(x,y) \vdash \exists x.\forall y.P(x,y)} \qquad \exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)}{\cfrac{\cfrac{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)}{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)} \; \forall I}{\vdash \exists x.\forall y.P(x,y) \to \forall y.\exists x.P(x,y)} \; {\to}I} \; \exists E$$

Gentzen Calculus

Exercise 1.b

! Proven !

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{\textit{Axiom}}}{\forall y.P(b,y) \vdash \forall y.P(b,y)}}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \forall y.P(b,y)}\ \boxed{\text{W}}}{\exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash P(b,a)}\ \boxed{\forall E}}{\cfrac{\text{\textit{Axiom}}}{\exists x.\forall y.P(x,y) \vdash \exists x.\forall y.P(x,y)} \qquad \exists x.\forall y.P(x,y), \forall y.P(b,y) \vdash \exists x.P(x,a)}\ \boxed{\exists I}}{\cfrac{\cfrac{\exists x.\forall y.P(x,y) \vdash \exists x.P(x,a)}{\exists x.\forall y.P(x,y) \vdash \forall y.\exists x.P(x,y)}\ \boxed{\forall I}}{\vdash \exists x.\forall y.P(x,y) \rightarrow \forall y.\exists x.P(x,y)}\ \boxed{\rightarrow I}}\ \boxed{\exists E}$$

Gentzen Calculus

Thanks for listening....

...Any questions ?