Exercises to the Lecture FSVT

Prof. Dr. Klaus Madlener sheet 6

Exercise 1:

Let $\lesssim \subseteq \text{Term}(F, V) \times \text{Term}(F, V)$ be defined as:

$$s \lesssim t \text{ iff. exists a substitution } \sigma \text{ with } t \equiv \sigma(s)$$
$$s \approx t \text{ iff. } s \lesssim t \text{ and } t \lesssim s$$
$$s < t \text{ iff. } s \lesssim t \text{ and } s \not\approx t$$

Show:

1. $<$ is strict part of a well-founded partial order. Which elements is this partial order defined on?
2. $s \approx t$ holds iff. a permutation ξ exists with $s \equiv \xi(t)$ (variable renaming).

Exercise 2:

This exercise is on an alternative specification of the integers INTEGER $= (sig, E)$ with

$$sig = (int, 0, succ, pred, add),$$
$$E = \{succ(pred(x)) = x, pred(succ(x)) = x, add(0, y) = y, add(succ(x), y) = succ(add(x, y))\}$$

1. Show, that $(\mathbb{Z}, 0, +1, -1, +)$ is initial in Alg(INTEGER).
2. Structurize this specification using the specification INT. Show that INTEGER is an enrichment of INT.
3. Extend INTEGER by a function `absolute` with the properties of the absolute value function on $\mathbb{Z}$. Show that this is an enrichment of INT.

Exercise 3:

Let INT2 be the specification of integers from example 7.9 of the lecture. We combine INT2 with BOOL and $((\{\}, \{<\}), E)$ to obtain a specification INT3, where

$E = \{< (0, \text{succ}(x)) = \text{true}, < (\text{pred}(x), 0) = \text{true}, < (0, \text{pred}(x)) = \text{false}, < (\text{succ}(x), 0) = \text{false}, < (\text{pred}(x), \text{pred}(y)) =< (x, y), < (\text{succ}(x), \text{succ}(y)) =< (x, y)\}$

1. Check, whether $T_{\text{INT3}} \mid_{\text{bool}} \cong$ Bool. Why would this be important? Hint: Look at $< (\text{succ}(\text{pred}(x)), \text{pred}(\text{succ}(y)))$.
2. Show that INT3 can not be fixed by additional equations.
3. Find further problems of INT3.

4. Make a suggestion for a specification INT4, such that $T_{\text{INT4}}|_{\text{int}} \cong \mathbb{Z}$, $T_{\text{INT4}}|_{\text{bool}} \cong$ Bool and $<$ is properly defined by its equations. Hint: Consider further function symbols.

Exercise 4:

Let $sig_1 = (\{\text{NAT}, \text{EVEN}\}, \{0, 1, S, f\}, \{0 :\to \text{NAT}, 1 :\to \text{EVEN}, S : \text{NAT} \to \text{NAT}, f : \text{NAT} \to \text{EVEN}\})$. Further, let the sig_1-Algebra $\mathfrak{A}_1$ be defined by:

$$A_{1,\text{NAT}} = \mathbb{N}, A_{1,\text{EVEN}} = 2\mathbb{N} \cup \{1\}, 0_{\mathfrak{A}_1} = 0, 1_{\mathfrak{A}_1} = 1, S_{\mathfrak{A}_1}(x) = x + 1, f_{\mathfrak{A}_1}(x) = \begin{cases} x, & \text{if } x \text{ even} \\ 1, & \text{else} \end{cases}$$

Prove:

1. There is no specification $spec_1 = (sig_1, E_1)$ with finite E_1, such that $T_{spec_1} \cong \mathfrak{A}_1$.
2. There is a specification $spec_2 = (sig_2, E_2)$ with $sig_1 \subseteq sig_2$, E_2 finite, such that $T_{spec_2}|_{sig_1} \cong \mathfrak{A}_1$.

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